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On the Zeros of Polynomials of Best Approximation

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Given a function f, uniform limit of analytic polynomials on a compact, regular set $E \subset \mathbb{C}^N$, we relate analytic extension properties of f to the location of the zeros of the best polynomial approximants to f in either the uniform norm on E or in appropriate L^q norms.

These results give multivariable versions of one-variable results due to Blatt–Saff, Pleśniak and Wójcik. © 1999 Academic Press

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0. INTRODUCTION

Let $E \subset \mathbb{C}^N$ be compact and regular (in the sense of pluripotential theory). Let $W(E)$ denote the closure in the uniform norm on E of $P(\mathbb{C}^N)$ (where $P(\mathbb{C}^N)$ denotes the analytic polynomials on $\mathbb{C}^N$). For $f \in W(E)$ we let $p_n(z)$ denote a best approximant to f from P_n, $n = 1, 2, 3, \ldots$, where P_n denotes the analytic polynomials of total degree $\leqslant n$. Given a positive Borel measure μ on E we let f_n denote the best approximant from P_n to f in $L^2(d\mu)$

In this paper we will study the relation between analytic extension properties of f and zeros of the sequences $\{p_n(z)\}$ or $\{f_n(z)\}$.

Let $V_E(z)$ denote the pluricomplex Green function of E (see (1.1) for the definition).

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For $R > 1$, we let

$$E_R = \{z \in \mathbb{C}^N \mid V_E(z) < \log R\}. \tag{0.1}$$

We will study analytic extension of f to open sets of the form E_R, i.e., does there exist F, analytic on E_R with $F/_E = f$?

In one variable, $V_E(z)$ is the Green function of $\mathbb{C}\backslash\hat{E}$ with a logarithmic pole at ∞ (and extended by zero on $\hat{E}$), where $\hat{E}$ denotes the polynomial convex hull of E. In the one variable case, there are extensive results due to S. N. Bernstein, H.-P. Blatt–E. Saff, P. Borwein, W. Pleśniak and A. Wójcik (see references). Roughly speaking, f has an analytic extension to E_R if and only if almost all zeros of $\{p_n\}$ or $\{f_n\}$ lie in $\mathbb{C}\backslash E_R$. If f is not analytic on E (i.e., does not have an analytic extension to a neighborhood of E) then every point on $\partial\hat{E}$ is a limit point of the zeros of $\{p_n\}$. The precise statements must discount zeros in the interior of $\hat{E}$, and must be modified if f is identically zero on a component of the interior of $\hat{E}$.

Specific one variable results (reformulated) are as follows

Theorem 0.1 (Wójcik [W]). *f has an analytic extension to E_R if the zeros of $\{p_n\}$ have no point of accumulation in E_R.*

Theorem 0.2 (Blatt–Saff [BS]). *Suppose f is not analytic on E and for some $z_0 \in \partial\hat{E}$, $f(z_0) \neq 0$. Then there exists a sequence of points $\{z_n\}$ with $\lim_{n\to\infty} z_n = z_0$ and $p_n(z_n) = 0$.*

Theorem 0.3 (Pleśniak [P]). *Let μ be a finite Borel measure on E which satisfies the Leja polynomial condition (see [P] for the definition). Then f has an analytic extension to E_R if the zeros of $\{f_n\}$ have no point of accumulation in E_R.*

In this paper we will give multivariable versions of these results. Theorem 3.5 and Corollary 3.6 generalize theorems 0.1 and 0.3. In Corollary 3.6, the hypothesis that μ satisfy the Bernstein–Markov condition is a less stringent on μ than requiring it to satisfy the Leja polynomial condition (see [Bl1]). Theorem 3.8 is a multivariable version of Theorem 0.2. In Theorem 3.8 an additional hypothesis is required (see (3.15) or (3.20)) on the set E.

We also prove (Theorem 2.1) an L^q analogue of a result on Tchebyshev polynomials ([Bl2, Theorem 3.1]; [Si3]). This is used in the proof of Corollary 3.6 but is of independent interest.

1. PRELIMINARIES

Let $E \subset \mathbb{C}^N$ be a compact set. Let V_E denote its pluricomplex extremal function, i.e.

$$V_E(z) = \sup \{u(z) : u \in \mathcal{L}, u/_E \leqslant 0\}, \tag{1.1}$$

where $\mathcal{L}$ denotes *the Lelong class* of plurisubharmonic functions satisfying

$$\sup_{z \in \mathbb{C}^N} u(z) - \log^+ |z| < \infty, \tag{1.2}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{C}^N$.

We shall assume that E is *regular*, i.e., the function V_E is continuous. This implies that E is not pluripolar. Recall that a set $E \subset \mathbb{C}^N$ is said to be pluripolar if for every $a \in E$ there is a neighborhood V of a and a plurisubharmonic function u on V such that $E \cap V \subset \{z \in V \mid u = -\infty\}$. Pluripolar sets have Lebesgue ($2n$–dimensional) measure zero [K, Cor. 2.9.10].

Let μ be a finite Borel measure on E such that the pair (E, μ) satisfies *the Bernstein–Markov condition* (BM), i.e., for any $\varepsilon > 0$ and q, $0 < q < \infty$, there exists $A = A(\varepsilon, q)$ such that

$$\|p\|_E \leqslant A(1+\varepsilon)^{\deg(p)} \|p\|_{\mu, q} \tag{1.3}$$

for all polynomials $p \in P(\mathbb{C}^N)$, where

$$\|p\|_{\mu, q} = \left(\int_E |p(z)|^q \, d\mu \right)^{1/q}. \tag{1.4}$$

It is known (see [Bl1, Remark 3.2]) that if μ satisfies (BM) for one exponent q, $0 < q < \infty$, then it satisfies (BM) for all q, $0 < q < \infty$.

Let $L^q_P(E, \mu)$, $1 \leqslant q < \infty$, denote the continuous functions on E that are limits of polynomials in the norm $\| \ \|_{\mu, q}$. Of course, $W(E) \subset L^q_P(E, \mu)$.

Let f be a continuous function on E. We denote by $p_n \in P_n$ and $f_n \in P_n$, respectively, polynomials of degree at most $n \in \mathbb{N}_0$ of best approximation in the uniform norm and the norm $\|\cdot\|_{\mu, q}$, respectively, i.e.

$$\|f - p_n\|_E = \inf \{ \|f - q_n\|_E, \ q_n \in P_n\}, \tag{1.5}$$

$$\|f - f_n\|_{\mu, q} = \inf \{ \|f - q_n\|_{\mu, q}, \ q_n \in P_n\}. \tag{1.6}$$

These best approximants are not necessarily unique.

From the Bernstein–Walsh–Siciak theorem [Si1, Theorem 10.3] and *the Bernstein–Walsh inequality* (BW)

$$|p(z)| \leqslant \|p\|_E (\exp V_E(z))^{\deg p}, \qquad \text{for} \quad z \in \mathbb{C}^N \quad \text{and} \tag{1.7}$$

for all polynomials $p \in P(\mathbb{C}^N)$ (see, eg., [Si2, 2.11] or [K, 5.1]) we conclude the following

Remark 1.1. (cf. [Si2, Corollary 8.6]) Let $E \subset \mathbb{C}^N$ be a compact, regular set and let μ be a Borel measure such that (E, μ) satisfies (BM). Let f be a continuous function on E and $R > 1$. If f admits an analytic extension onto E_R, denoted F, then the sequences of polynomials of best approximation $\{p_n\}$ and $\{f_n\}$ in the norms $\|\cdot\|_E$ and $\|\cdot\|_{\mu, q}$ are uniformly bounded on compact subsets of E_R and, for $1 < r < R$, we have

$$\limsup_{n \to \infty} \|F - p_n\|_{E_r}^{1/n} \leqslant \frac{r}{R} \tag{1.8}$$

and

$$\limsup_{n \to \infty} \|F - f_n\|_{E_r}^{1/n} \leqslant \frac{r}{R}. \tag{1.9}$$

We shall denote by $\hat{r}_n$ the homogeneous part of degree n of the polynomial $r_n(z) = \sum_{|\alpha| \leqslant n} a_\alpha z^\alpha$, i.e., $\hat{r}_n(z) = \sum_{|\alpha| = n} a_\alpha z^\alpha$. If $\deg r_n < n$ we put $\hat{r}_n(z) \equiv 0$.

For a homogeneous polynomial $h_n(z) = \sum_{|\alpha| = n} a_\alpha z^\alpha$ we define *the Tchebyshev polynomials* associated with h_n and the norms $\|\cdot\|_E$, $\|\cdot\|_{\mu, q}$, respectively, by

$$Tch_E\, h_n(z) = h_n(z) - r_{n-1}, \tag{1.10}$$

$$Tch_{\mu, q}\, h_n(z) = h_n(z) - s_{n-1}, \tag{1.11}$$

where $r_{n-1}, s_{n-1} \in P_{n-1}$ are polynomials of best approximation to h_n in the norms $\|\cdot\|_E$ and $\|\cdot\|_{\mu, q}$, respectively, i.e.

$$\|h_n(z) - r_{n-1}\|_E = \inf\{\|h_n(z) - p_{n-1}\|_E,\ p_{n-1} \in P_{n-1}\},$$

$$\|h_n(z) - s_{n-1}\|_{\mu, q} = \inf\{\|h_n(z) - p_{n-1}\|_{\mu, q},\ p_{n-1} \in P_{n-1}\}.$$

In general, $Tch_E h_n$ and $Tch_{\mu, q} h_n$ need not be unique (except for $q = 2$, since $L^2_P(E, \mu)$ is a Hilbert space, thus s_{n-1} is the orthogonal projection of h_n on P_{n-1}), however the norms $\|Tch_E h_n\|_E$ and $\|Tch_{\mu, q} h_n\|_{\mu, q}$ are unambiguously defined.

LEMMA 1.2 (see also [Sz, Lemma 5.2]). *Let* (E, μ) *satisfy the Bernstein–Markov condition. Let* $\{h_n\}_{n=1,2,3,\cdots}$ *be a sequence of homogeneous polynomials of degree n. Let q satisfy* $1 \leqslant q < \infty$. *Then*

$$\limsup_{n\to\infty} \|Tch_E h_n\|_E^{1/n} = \limsup_{n\to\infty} \|Tch_E h_n\|_{\mu,q}^{1/n} = \limsup_{n\to\infty} \|Tch_{\mu,q} h_n\|_E^{1/n}$$
$$= \limsup_{n\to\infty} \|Tch_{\mu,q} h_n\|_{\mu,q}^{1/n}.$$

Proof. From (BM) we have, for every $\varepsilon > 0$

$$\|Tch_{\mu,q} h_n\|_E \leqslant A(1+\varepsilon)^n \|Tch_{\mu,q} h_n\|_{\mu,q}. \tag{1.12}$$

Since $Tch_{\mu,q} h_n$ is a competitor for $Tch_E h_n$ we have

$$\|Tch_E h_n\|_E \leqslant \|Tch_{\mu,q} h_n\|_E. \tag{1.13}$$

Similarly

$$\|Tch_{\mu,q} h_n\|_{\mu,q} \leqslant \|Tch_E h_n\|_{\mu,q}. \tag{1.14}$$

Since $\mu(E) < +\infty$ we have

$$\|Tch_E h_n\|_{\mu,q} \leqslant \mu(E)^{1/q} \|Tch_E h_n\|_E \tag{1.15}$$

Now, (1.12) and (1.13) imply

$$\limsup_{n\to\infty} \|Tch_E h_n\|_E^{1/n} \leqslant \limsup_{n\to\infty} \|Tch_{\mu,q} h_n\|_E^{1/n} \leqslant \limsup_{n\to\infty} \|Tch_{\mu,q} h_n\|_{\mu,q}^{1/n}.$$

Also, (1.14) and (1.15) imply

$$\limsup_{n\to\infty} \|Tch_{\mu,q} h_n\|_{\mu,q}^{1/n} \leqslant \limsup_{n\to\infty} \|Tch_E h_n\|_{\mu,q}^{1/n} \leqslant \limsup_{n\to\infty} \|Tch_E h_n\|_E^{1/n}.$$

The result follows. ∎

For $u \in \mathscr{L}$ we define *the Robin function* associated with u by

$$\rho_u([z]) = \limsup_{\substack{|\lambda|\to\infty \\ \lambda\in\mathbb{C}}} u(\lambda z) - \log^+ |\lambda z|, \qquad \text{for} \quad z \in \mathbb{C}^N \setminus \{0\}, \tag{1.16}$$

where $[\,\cdot\,]: \mathbb{C}^N \setminus \{0\} \ni z \to [z] \in \mathbb{P}^{N-1}$ denotes the natural map and $\mathbb{P}^{N-1}$ denotes complex projective $(N-1)$-space.

It is seen that if $p_n \in P_n$ then $u_n(z) := 1/n \log |p_n(z)| \in \mathscr{L}$ and

$$\rho_{u_n}([z]) = \frac{1}{n} \log |\hat{p}_n(z)| - \log |z|, \qquad \text{for} \quad z \neq 0. \tag{1.17}$$

If $\deg p_n < n$ then $\hat{p}_n(z) \equiv 0$ and we put $\rho_{u_n} \equiv -\infty$.

The Robin function of a compact, regular set E, denoted $\rho_E([z])$ is defined as the Robin function of $V_E(z)$.

2. AN APPLICATION OF THE ROBIN FUNCTION TO POLYNOMIAL APPROXIMATION—THE CASE OF L^q NORMS

We prove a counterpart of [Bl2, Theorem 3.1] in the case of the polynomial approximation in L^q-norms.

THEOREM 2.1. *Let $E \subset \mathbb{C}^N$ be a compact, regular set and let μ be a Borel measure such that the pair (E, μ) satisfies (BM). Fix q, $1 \leqslant q < \infty$. Let $f \in L^q_P(E, \mu)$ and $R > 1$. The following conditions are equivalent*

$$f \text{ extends analytically to } E_R \tag{2.1}$$

$$\limsup_{n \to \infty} \| f - f_n \|_{\mu, q}^{1/n} \leqslant \frac{1}{R} \tag{2.2}$$

$$\limsup_{n \to \infty} \| Tch_{\mu, q} \hat{f}_n \|_E^{1/n} \leqslant \frac{1}{R} \tag{2.3}$$

$$\limsup_{n \to \infty} \| Tch_{\mu, q} \hat{f}_n \|_{\mu, q}^{1/n} \leqslant \frac{1}{R} \tag{2.4}$$

$$\limsup_{n \to \infty} \frac{1}{n} \log |\hat{f}_n(z)| - \log |z| \leqslant \rho_E([z]) - \log R, \qquad \text{for all} \quad z \neq 0, \tag{2.5}$$

where $\{f_n\}$ is a sequence of polynomials of best approximation in the norm $\| \cdot \|_{\mu, q}$.

The proof of the theorem is analogous to the proof of [Bl2, Theorem 3.1]

$$(2.1) \Rightarrow (2.2) \Rightarrow (2.4) \Rightarrow (2.2) \Rightarrow (2.1)$$

$$(2.3) \Leftrightarrow (2.4)$$

$$(2.3) \Rightarrow (2.5) \Rightarrow (2.4).$$

Proof. Assume (2.1). By the Bernstein–Walsh–Siciak theorem we get

$$\limsup_{n \to \infty} \| f - p_n \|_E^{1/n} \leqslant \frac{1}{R}, \tag{2.6}$$

where $\{p_n\}$ is a sequence of polynomials of best uniform approximation to f on E. Since

$$\|f-f_n\|_{\mu,q} \leqslant \|f-p_n\|_{\mu,q} \leqslant (\mu(E))^{1/q} \|f-p_n\|_E, \tag{2.7}$$

we have

$$\limsup_{n\to\infty} \|f-f_n\|_{\mu,q}^{1/n} \leqslant \frac{1}{R}, \tag{2.8}$$

i.e., (2.2) holds.

Next, by the definition of the Tchebyshev polynomial $Tch_{\mu,q}\hat{f}_n$ and by (1.6), we get

$$\begin{aligned} \|Tch_{\mu,q}\hat{f}_n\|_{\mu,q} &\leqslant \|f_n-f_{n-1}\|_{\mu,q} \\ &\leqslant \|f_n-f\|_{\mu,q} + \|f-f_{n-1}\|_{\mu,q} \\ &\leqslant 2\,\|f-f_{n-1}\|_{\mu,q} \end{aligned} \tag{2.9}$$

Thus, by (2.2),

$$\limsup_{n\to\infty} \|Tch_{\mu,q}\hat{f}_n\|_{\mu,q}^{1/n} \leqslant \limsup_{n\to\infty} \|f-f_n\|_{\mu,q}^{1/n} \leqslant \frac{1}{R},$$

and (2.4) follows.

Since the pair (E,μ) satisfies (BM), by Lemma 1.2, the conditions (2.3) and (2.4) are equivalent.

Assume (2.4). By the definition of the Tchebyshev polynomial $Tch_{\mu,q}\hat{f}_{n+1}$, we have

$$\begin{aligned} \|f-f_n\|_{\mu,q} &\leqslant \|f-(f_{n+1}-Tch_{\mu,q}\hat{f}_{n+1})\|_{\mu,q} \\ &\leqslant \|f-f_{n+1}\|_{\mu,q} + \|Tch_{\mu,q}\hat{f}_{n+1})\|_{\mu,q}. \end{aligned} \tag{2.10}$$

Since $\|f-f_n\|_{\mu,q} \geqslant \|f-f_{n+1}\|_{\mu,q}$ and $\lim_{n\to\infty} \|f-f_n\|_{\mu,q}=0$, we have, by (2.4),

$$\limsup_{n\to\infty} \|f-f_n\|_{\mu,q}^{1/n} \leqslant \frac{1}{R}, \tag{2.11}$$

i.e., (2.2) holds.

Next, suppose that (2.2) holds. Fix r, $1<r<R$, and then fix $\varepsilon>0$ and ρ such that $(1+\varepsilon)\,r<\rho<R$. By (2.2), there exists $n_0=n_0(\rho)$ such that

$$\|f-f_n\|_{\mu,q} \leqslant \frac{1}{\rho^n} \qquad \text{for} \quad n \geqslant n_0\,. \tag{2.12}$$

By (BW),

$$\|f_{n+1}-f_n\|_{\overline{E_r}} \leqslant r^{n+1}\|f_{n+1}-f_n\|_E. \tag{2.13}$$

Next, by (BM), there exists $A_\varepsilon > 0$ such that

$$\|f_{n+1}-f_n\|_E \leqslant A_\varepsilon(1+\varepsilon)^{n+1}\ \|f_{n+1}-f_n\|_{\mu,q}, \text{ for all } n. \tag{2.14}$$

Since

$$\|f_{n+1}-f_n\|_{\mu,q} \leqslant \|f_{n+1}-f\|_{\mu,q} + \|f_n - f\|_{\mu,q} \leqslant 2\|f-f_n\|_{\mu,q}, \tag{2.15}$$

we get, for $n \geqslant n_0$, $\|f_{n+1}-f_n\|_{\overline{E_r}} \leqslant 2A_\varepsilon\rho((1+\varepsilon)r/\rho)^{n+1}$. Thus, for all $M, n \geqslant n_0$, we have

$$\sum_{k=n}^{M} \|f_{k+1}-f_k\|_{\overline{E_r}} \leqslant C\left(\frac{(1+\varepsilon)\,r}{\rho}\right)^{n+1}, \tag{2.16}$$

where $C = 2A_\varepsilon\rho^2(\rho-(1+\varepsilon)\,r)^{-1}$. Since $(1+\varepsilon)\,r < \rho$, the series

$$f_0 + \sum_{k=0}^{\infty} f_{k+1} - f_k$$

converges uniformly on $\overline{E_r}$. Since $r < R$ has been chosen arbitrary, we conclude that f extends analytically to E_R. i.e., (2.1) follows.

Suppose that (2.3) holds. Fix r such that $1 < r < R$. Then for $n \geqslant n_1(r)$ we have $\|Tch_{\mu,q}\hat{f}_n\|_E^{1/n} \leqslant 1/r$ or

$$\log r + \frac{1}{n}\log|Tch_{\mu,q}\hat{f}_n(z)| \leqslant 0, \qquad \text{for all} \quad z \in E. \tag{2.17}$$

Hence, by the definition of the pluricomplex extremal function V_E (see (1.1)), we have

$$\log r + \frac{1}{n}\log|Tch_{\mu,q}\hat{f}_n(z)| \leqslant V_E(z), \qquad \text{for all} \quad z \in \mathbb{C}^N. \tag{2.19}$$

Taking the Robin function of the both sides of (2.19) gives

$$\log r + \frac{1}{n}\log|\hat{f}_n(z)| - \log|z| \leqslant \rho_E([z]), \qquad \text{for all} \quad z \in \mathbb{C}^N\backslash\{0\}. \tag{2.20}$$

Since it holds for all $n \geqslant n_1(r)$ and all $r < R$ we get (2.5).

It remains to prove (2.5) $\Rightarrow$ (2.4). This follows from the following

LEMMA 2.2 (cf. [Bl2, Theorem 3.2]). *Let $E \subset \mathbb{C}^N$ be a compact, regular set and let μ be a Borel measure such that (E, μ) satisfies* (BM). *Let q satisfy $1 \leqslant q < \infty$. Let h_n be a sequence of homogeneous polynomials satisfying $\deg h_n = n$ or $h_n(z) \equiv 0$, for all $n \in \mathbb{N}_0$. Let $R > 1$. If*

$$\limsup_{n \to \infty} \frac{1}{n} \log |h_n(z)| - \log |z| \leqslant \rho_E([z]) - \log R, \qquad \text{for all} \quad z \in \mathbb{C}^N \setminus \{0\} \tag{2.21}$$

then

$$\limsup_{n \to \infty} \| Tch_{\mu, q} h_n \|_{\mu, q}^{1/n} \leqslant \frac{1}{R}. \tag{2.22}$$

Recall (1.13) that if $h_n(z) \equiv 0$ we put $1/n \log |h_n(z)| - \log |z| \equiv -\infty$.

Proof of of Lemma 2.2. From [Bl2, Theorem 3.1] (2.21) we have that

$$\limsup_{n \to \infty} \| Tch_E h_n \|_E^{1/n} \leqslant \frac{1}{R}$$

and the result now follows from Lemma 1.2. ∎

Putting $h_n := \hat{f}_n$ in the above lemma gives the implication (2.5) ⇒ (2.4). This completes the proof of Theorem 2.1.

3. ZEROS OF POLYNOMIALS OF BEST APPROXIMATION

Let E be a compact, regular subset of $\mathbb{C}^N$, $f \in W(E)$ and $\{p_n\}$ a sequence of best approximants to f in the uniform norm on E. We will relate the location of the zeros of $\{p_n\}$ to the analytic extension properties of f.

First we consider

$$v(z) := \left(\limsup_{n \to \infty} \frac{1}{n} \log |p_n(z)| \right)^* \tag{3.1}$$

where $(\quad)^*$ denotes upper semi-continuous regularization. (Recall that $(g(z))^* = \overline{\lim}_{\xi \to z}\, g(\xi)$.)

The sequence $\{p_n\}$ is uniformly bounded on E. Since E is non pluri-polar, $\{p_n(z)\}$ is locally bounded from above on $\mathbb{C}^N$ by (BW), and so $v \in \mathscr{L}$ [K, prop. 5.2.1].

Let

$$Z := \{ z \in \mathbb{C}^N \mid v(z) \leqslant 0 \} \tag{3.2}$$

and we let int(Z) denote the interior of the set Z. Let $R > 1$. We have

LEMMA 3.1. *The function f extends to a holomorphic function on E_R if and only if* $\operatorname{int}(Z) \supset E_R$.

Proof. Suppose that $E_R \subset \operatorname{int}(Z)$. Let $1 < r < R$. Then $\overline{E_r} \subset \operatorname{int}(Z)$. Since

$$\limsup_{n \to \infty} \frac{1}{n} \log |p_n(z)| \leqslant 0 \qquad \text{on} \quad \overline{E_r}$$

we have by Hartogs lemma, given $\varepsilon > 0$, there exists $n_0(\varepsilon)$

$$\frac{1}{n} \log |p_n(z)| \leqslant \varepsilon, \qquad \text{for} \quad n \geqslant n_0(\varepsilon) \quad \text{and} \quad z \in \overline{E_r} \tag{3.3}$$

and

$$\frac{1}{n} \log |p_n(z)| \leqslant \varepsilon + V_{\overline{E_r}}(z) \qquad \text{for} \quad z \in \mathbb{C}^N \quad \text{and} \quad n \geqslant n_0 \,. \tag{3.4}$$

Taking the Robin function of both sides of the above inequality gives

$$\frac{1}{n} \log |\hat{p}_n(z)| - \log |z| \leqslant \varepsilon + \rho_{\overline{E_r}}([z]) \qquad \text{for} \quad n \geqslant n_0 \,. \tag{3.5}$$

Recall that since E is regular,

$$V_{\overline{E_r}}(z) = \max\{ V_E(z) - \log r, 0\} \tag{3.6}$$

and

$$\rho_{\overline{E_r}}([z]) = \rho_E([z]) - \log r, \qquad \text{for all} \quad z \neq 0. \tag{3.7}$$

Using (3.5) and (3.7) gives

$$\frac{1}{n} \log |\hat{p}_n(z)| - \log |z| \leqslant \varepsilon + \rho_E([z]) - \log r \qquad \text{for} \quad n \geqslant n_0. \tag{3.8}$$

Hence, for all $\varepsilon > 0$, we have

$$\limsup_{n \to \infty} \frac{1}{n} \log |\hat{p}_n(z)| - \log |z| \leqslant \varepsilon + \rho_E([z]) - \log r \tag{3.9}$$

This implies

$$\limsup_{n \to \infty} \frac{1}{n} \log |\hat{p}_n(z)| - \log |z| \leqslant \rho_E([z]) - \log r \tag{3.10}$$

Then, by [Bl2, Theorem 3.1] f extends analytically to E_r. Since (3.10) is valid for any $r<R$, the function f extends analytically to E_R.

Conversely, if f extends analytically to E_R, by Remark 1.1, the sequence $\{p_n\}$ is uniformly bounded on $\overline{E_r}$, for all $r<R$. Thus, $v \leqslant 0$ on $\overline{E_r}$ and since $\bigcup_{r<R} \overline{E_r} \subset E_R$, we have $v \leqslant 0$ on E_R and so $E_R \subset \operatorname{int}(Z)$, since E_R is open (cf. 1.1). ∎

Let $E \subset \mathbb{C}^N$ be a compact, regular set and let μ be a finite Borel measure such that (E, μ) satisfies (BM). Let q satisfy $1 \leqslant q < \infty$. Let $\{f_n\}$ be a sequence of polynomials of best approximation in the norm $\|\cdot\|_{\mu, q}$ to a function $f \in L^q_P(E, \mu)$. We put

$$v_\mu(z) := \left(\limsup_{n \to \infty} \frac{1}{n} \log |f_n(z)| \right)^* \tag{3.11}$$

The sequence $\{f_n(z)\}$ is uniformly bounded in $L^q(E, \mu)$. Using (BM) we conclude that $\limsup_{n \to \infty} \|f_n\|_E^{1/n} \leqslant 1$ so $v_\mu \in \mathscr{L}$ (by [Bl1, Lemma 3.2] and [K, prop. 5.2.1]).

Let $Z_\mu = \{z \in \mathbb{C}^N : v_\mu(z) \leqslant 0\}$. Let $\operatorname{int}(Z_\mu)$ denote the interior of the set Z_μ.

LEMMA 3.2. *Let $f \in L^q_P(E, \mu)$. Let $R > 1$. The function f extends analytically to E_R if and only if* $\operatorname{int}(Z_\mu) \supset E_R$.

The proof of the above lemma is analogous to the proof of Lemma 3.1. To prove that $E_R \subset \operatorname{int}(Z_\mu)$ implies that f extends analytically to E_R, one should repeat (3.3)–(3.10) putting $\{f_n\}$ instead of $\{p_n\}$ and use Theorem 2.1 in (3.10) instead of [Bl2, Theorem 3.1]. The converse implication follows from Remark 1.1. ∎

LEMMA 3.3. (i) *$Z \supset E$ and $Z_\mu \supset E$.*

(ii) *Let $f \in W(E)$ not be analytic on E. Then $\partial Z \cap E \neq \emptyset$. Similarly for $f \in L^q_P(E, \mu)$ and f not analytic on E, then $\partial Z_\mu \cap E \neq \emptyset$.*

(iii) *Let $f \in W(E)$ have an analytic extension to E_R (for some $R > 1$) but not to E_s for any $s > R$. Then $\partial E_R \cap \partial(\operatorname{int}(Z)) \neq \emptyset$ and $\partial E_R \cap \partial(\operatorname{int}(Z_\mu)) \neq \emptyset$.*

Proof. (i) To show that $Z \supset E$ we must show that $v \leqslant 0$ on E.

The sequence $\{p_n(z)\}$ is uniformly bounded on E so $\limsup_{n \to \infty} 1/n \log |p_n(z)| \leqslant 0$ on E. Since negligible sets are pluripolar [K, Cor 4.6.2] we have $v \leqslant 0$ on $E \setminus N$, where N is pluripolar, so $v \leqslant V^*_{E \setminus N}$. But by [K, Cor 5.2.5] $V^*_{E \setminus N} = V_E$ and $V_E \equiv 0$ on E.

The proof that $Z_\mu \supset E$ is similar.

(ii) That $\partial Z \cap E \neq \emptyset$ follows from (i) and Lemma 3.1. That $\partial Z_\mu \cap E \neq \emptyset$ follows from (i) and Lemma 3.2.

(iii) This follows from Lemmas 3.1 and 3.2. ∎

EXAMPLE 3.4. Let E be a compact, regular subset of $\mathbb{C}$ and let $f \in W(E)$ not be analytic on E. Then it follows from ([BS, Theorem 2.1 and Lemma 4.2]) that v (defined by (3.1)) is the Green function of $\mathbb{C} \backslash \hat{E}$ and that $Z = \hat{E}$.

In the multivariable case we will give an example of $f \in W(E)$, not analytic on E but where $Z \neq E$ (E will be polynomially convex so that $E = \hat{E}$).

Let E be the unit ball in $\mathbb{C}^2$. $E = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 \leqslant 1\}$ and let $f = f(z_1)$ be a function continuous on $\Delta_1 = \{z_1 \in \mathbb{C} : |z_1| \leqslant 1\}$ analytic on $\operatorname{int}(\Delta_1)$ and not analytic on Δ_1. Let $p_n(z_1)$ denote the best polynomial approximant of degree $\leqslant n$ to f on Δ_1. Then $p_n(z_1)$ is a best approximant of total degree $\leqslant n$ to f considered as a function on E. Using the above quoted result of Blatt-Saff it follows that

$$\lim_{n \to \infty} \frac{1}{n} \log |p_n(z_1)| = \log |z_1| \qquad \text{for} \quad |z_1| \geqslant 1$$

so it follows that $Z = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leqslant 1\}$. Note that $\partial Z \cap E$ does not contain either the topological or Silov boundary of E, in contrast, to the one-variable case.

THEOREM 3.5. *Let f be holomorphic on E_R and let $\widetilde{E}_R$ be the union of those components of E_R, where f is not identically equal to zero. Let $\{p_n\}$ be a sequence of best uniform approximants to f on E. Let $z_0 \in \partial \widetilde{E}_R \cap \partial(\operatorname{int}(Z))$. Then there exists a sequence of points $\{z_n\}$ with $\lim_{n \to \infty} z_n = z_0$ and $p_n(z_n) = 0$.*

Proof. The proof is by contradiction. Suppose that z_0 is not such a limit point. Then there is a ball B centred at z_0 such that $p_n \neq 0$ on B for $n \geqslant n_1$. Chose an analytic branch of $p_n^{1/n}$ on B. For some constant $M_1 > 0$ we have $v(z) \leqslant M_1$, for all $z \in \bar{B}$. Hence

$$|p_n^{1/n}(z)| = \exp\left(\frac{1}{n} \log |p_n(z)|\right) \leqslant \exp 2M_1 \qquad \text{for all} \quad n \geqslant n_2(M_1) \tag{3.12}$$

and the sequence $\{p_n^{1/n}\}$ is a uniformly bounded sequence of analytic functions on B. Now $B \not\subseteq \operatorname{int}(Z)$, so there is a point $z_1 \in B$ where $\limsup_{n \to \infty} 1/n \log |p_n(z_1)| > 0$ since $\limsup_{n \to \infty} 1/n \log |p_n(z)| = v(z)$, except possibly on a pluripolar set.

Chose a subsequence $J \subset \mathbb{N}_0$ such that

$$\lim_{n \in J} \frac{1}{n} \log |p_n(z_1)| > 0 \tag{3.13}$$

Let J_1 be a subsequence of J such that the uniformly bounded sequence of analytic functions $\{p_n^{1/n}(z)\}_{n \in J_1}$ converges uniformly on compact subsets of B to an analytic function, denoted $g(z)$. Then

$$\log |g(z)| = \lim_{n \in J_1} \frac{1}{n} \log |p_n(z)|, \qquad \text{for} \quad z \in B, \tag{3.14}$$

so $|g(z_1)| > 1$ and $|g(z)| \leqslant 1$ for $z \in \text{int}(Z) \cap B$.

Thus g is not constant on B and so by the maximum modulus principle $|g(z)| < 1$ on $\text{int}(Z) \cap B$. This implies that $\lim_{n \in J_1} |p_n(z)|^{1/n} < 1$, for $z \in \text{int}(Z) \cap B$, and so $\lim_{n \in J_1} |p_n(z)| = 0$. But on $B \cap E_R$, the sequence $\{p_n\}$ converges to f uniformly on compact subsets. Hence $f \equiv 0$ on $B \cap \widetilde{E}_R$, which contradicts the assumption that f is not identically zero on any component of $\widetilde{E}_R$. ∎

The above proof is based on the proof of Theorem 2.2 in [BS]. See also [Wa, theorem 2].

Let $E \subset \mathbb{C}^N$ be a compact, regular set and let μ be a finite Borel measure such that (E, μ) satisfies (BM). Let q satisfy $1 \leqslant q < \infty$.

Proceeding in the same manner as in the proof of Theorem 3.5, one easily proves

COROLLARY 3.6. *Let f be holomorphic on E_R and let $\widetilde{E}_R$ be the union of connected components of E_R where f is not identically equal to zero. Let $\{f_n\}$ be a sequence of best approximants to f on E in the norm $\|\cdot\|_{\mu,q}$. Let $z_0 \in \partial \widetilde{E}_R \cap \partial(\text{int}(Z_\mu))$. Then there exists a sequence of points $\{z_n\}$ such that $\lim_{n \to \infty} z_n = z_0$ and $f_n(z_n) = 0$.*

Remark 3.7. Let f be holomorphic on E_R but not on E_s (for any $s > R$). Let $\{p_n\}$ be a sequence of best uniform approximants to f on E. Let $z_0 \in \partial E_R \cap \partial(\text{int}(Z))$ and let α be a complex number such that there is a connected component of E_R with z_0 in its closure and f is not identically equal to α on that component. Then there exists a sequence of points $z_n(\alpha)$ such that $\lim_{n \to \infty} z_n(\alpha) = z_0$ and $p_n(z_n(\alpha)) = 0$.

This is because $p_n - \alpha$ is a best approximant from P_n to $f - \alpha$. This, in turn, shows that the sequence of best approximants $\{p_n\}$ have "the behaviour of an essential singularity" at every point of $\partial E_R \cap \partial(\text{int}(Z))$. Precisely, for every point $z_0 \in \partial E_R \cap \partial(\text{int}(Z))$ and every neighborhood N of z_0 the values $\bigcup_{n=1}^{\infty} p_n(V)$ are equal to $\mathbb{C}$ or omit at most one complex number. In particular the sequence $\{p_n\}$ does not converge uniformly on any neighborhood of z_0 although the function f may have an analytic extension to a neighborhood of z_0.

Similarly, the sequence of best approximants $\{f_n\}$ has "the behavior of an essential singularity" at every point of $\partial E_R \cap \partial(\mathrm{int}(Z_\mu))$.

We now turn to the case that f is not analytic on E. We will give a multivariable version of ([BS, Theorem 2.2]). That result is valid for E a regular, compact, polynomially convex subset of $\mathbb{C}$ whereas our generalization requires an additional hypothesis on E. (We conjecture Theorem 3.8 to be valid without this additional hypothesis).

We introduce:

$$\text{For all } z \in E \text{ and any ball } B \text{ centered at } z, \text{ there is a connected component } E' \text{ of } \bar{B} \cap E \text{ which is not pluripolar.} \tag{3.15}$$

THEOREM 3.8. *Let $f \in W(E)$ and suppose f is not analytic on E. Assume that E satisfies* (3.15). *Let $z_0 \in \partial Z \cap E$ be such that $f(z_0) \neq 0$. Then there exists a sequence of points $\{z_n\}$, such that* $\lim_{n \to \infty} z_n = z_0$ *and $p_n(z_n) = 0$, for $n = 1, 2, 3, \ldots$.*

Proof. Note that for $f \in W(E)$ if f is not analytic on E then, by Lemma 3.3, $\partial Z \cap E \neq \emptyset$. The proof is by contradiction. Suppose that z_0 is not such a limit point. Proceeding as in the proof of Theorem 3.5 we may assume there is a ball B, with center z_0, sufficiently small radius and an integer n_1 such that

$$|f(z) - f(z_0)| < \left|\frac{f(z_0)}{4}\right| \qquad \text{for} \quad z \in E \cap \bar{B} \tag{3.16}$$

and

$$|p_n(z) - f(z_0)| < \left|\frac{f(z_0)}{2}\right| \qquad \text{for} \quad z \in E \cap \bar{B} \tag{3.17}$$

for $n \geqslant n_1$.

Furthermore we may assume $p_n(z)$ has no zero on B for $n \geqslant n_1$. For $n \geqslant n_1$ we choose an analytic branch of $\log p_n(z)$ on B.

As in the proof of Theorem 3.5, we may assume there is a subsequence $J_1 \subset \mathbb{N}$ such that

$$g_1(z) := \lim_{n \in J_1} \exp\left(\frac{1}{n} \log p_n(z)\right) \tag{3.18}$$

is analytic and non constant on B.

Let $\log_1$ be an analytic branch of the logarithm function on the set

$$G=\left\{\tau\in\mathbb{C}\,|\,|\tau-f(z_0)|<\left|\frac{f(z_0)}{2}\right|\right\}.$$

Then $\log_1(p_n(z))$ is defined for $z\in E\cap\bar{B}$ and $n\geqslant n_1$. Now

$$\frac{1}{2\pi i}\left[\log_1(p_n(z))-\log(p_n(z))\right]$$

is continuous on $E\cap\bar{B}$ and integer-valued. Hence it must be constant on E' (by hypothesis (3.15) this is a connected component of $E\cap\bar{B}$). Let $t_n\in\mathbb{Z}$ denote its value.

We then consider the functions $\log(p_n(z)+2\pi i t_n)$ and we may choose a subsequence $J_2\subset J_1$ so that

$$g_2(z)\ :=\lim_{n\in J_2}\exp\left(\frac{1}{n}(\log(p_n(z)\right)+2\pi i t_n))\tag{3.19}$$

is analytic on B.

But $Im(\log(p_n(z)+2\pi i t_n))$ is bounded on E' since $Im(\log_1(\tau))$ is bounded on G. Thus $g_2(z)=1$ for all $z\in E'$ and since E' is not pluripolar, $g_2(z)=1$ for all $z\in B$. But $g_1(z)=cg_2(z)$ for some constant c, $|c|=1$. Hence $g_1(z)$ is constant on B. This contradiction establishes the result. ∎

Remark 3.9. Note that Theorem 3.8 is also valid under the hypothesis (3.20) below rather than (3.15):

For all $z\in E$ and all balls B centered at z, $E\cap B$ is not contained in a proper real-analytic subvariety of B. (3.20)

This is because $E\cap B\subset\{z\in B\,|\,|g_1(z)|=1\}$ and g_1 is non-constant, analytic on B.

Under the hypothesis of Theorem 3.8, we may conclude the following:

Remark 3.10. Given any complex number α and $z_0\in\partial Z\cap E$ with $f(z_0)\neq\alpha$ there exists a sequence of points $z_n(\alpha)$ satisfying $\lim_{n\to\infty}z_n(\alpha)=z_0$ and $p_n(z_n(\alpha))=\alpha$ (see Remark 3.7).

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